Spectra and symmetric eigentensors of the Lichnerowicz Laplacian on $P^n(\mathbb{C})$

Mohamed Boucetta *

Abstract. We compute the eigenvalues with multiplicities of the Lichnerowicz Laplacian acting on the space of complex symmetric covariant tensor fields on the complex projective space $P^n(\mathbb{C})$. The spaces of symmetric eigentensors are explicitly given.

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1 Introduction

Let (M,g) be a Riemannian n-manifold and D its Levi-Civita connection. For any $p \in \mathbb{N}$, we shall denote by $\Gamma(\otimes^p T^*M, \mathbb{C})$, $\Omega^p(M, \mathbb{C})$ and $\mathcal{S}^p(M, \mathbb{C})$ the space of complex covariant p-tensor fields on M, the space of complex differential p-forms on M and the space of complex symmetric covariant p-tensor fields on M, respectively. Note that $\Gamma(\otimes^0 T^*M, \mathbb{C}) = \Omega^0(M, \mathbb{C}) = \mathcal{S}^0(M, \mathbb{C}) = C^\infty(M, \mathbb{C})$. We put

$$\Omega(M, \mathbb{C}) = \bigoplus_{p=0}^{n} \Omega^p(M, \mathbb{C}) \quad \text{and} \quad \mathcal{S}(M, \mathbb{C}) = \bigoplus_{p \geq 0} \mathcal{S}^p(M, \mathbb{C}).$$

If (M,g) is Kählerian, i.e., there exists a complex structure J on M such that $DJ = 0$ and g is Hermitian with respect to J. The complex structure J defines a bigraduation

$$\Omega^p(M, \mathbb{C}) = \bigoplus_{r+q=p} \Omega^{r,q}(M) \quad \text{and} \quad \mathcal{S}^p(M, \mathbb{C}) = \bigoplus_{r+q=p} \mathcal{S}^{r,q}(M).$$

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We consider the Lichnerowicz Laplacian $\Delta_M : \Gamma(\otimes^* T^*M, \mathbb{C}) \longrightarrow \Gamma(\otimes^* T^*M, \mathbb{C})$ introduced by Lichnerowicz in [16] pp. 26. It is a second order differential operator, self-adjoint, elliptic and respects the symmetries of tensor fields. In particular, Δ_M leaves invariant $\mathcal{S}(M,\ \mathbb{C})$ and the restriction of Δ_M to $\Omega(M, \mathbb{C})$ coincides with the Hodge-de Rham Laplacian. Moreover, the Lichnerowicz Laplacian respects the bigraduation induced by Kählerian structures.

The Lichnerowicz Laplacian acting on symmetric covariant tensor fields is of fundamental importance in mathematical physics (see for instance [10], [21] and [23]). Note also that the Lichnerowicz Laplacian acting on symmetric covariant 2-tensor fields appears in many problems in Riemannian geometry (see [3], [5], [20]...).

On a compact Riemannian manifold, the Lichnerowicz Laplacian Δ_M has discrete eigenvalues with finite multiplicities. For a given compact Riemannian manifold, it may be an interesting problem to determine explicitly the eigenvalues and the eigentensors of Δ_M on M.

Let us enumerate the cases where the spectra of Δ_M was computed:

1. Δ_M acting on $C^\infty(M, \mathbb{C})$: M is either flat torus or Klein bottles [4], M is a Hopf manifold [1];

2. Δ_M acting on $\Omega(M, \mathbb{C})$: $M = S^n$ or $P^n(\mathbb{C})$ [11] and [12], $M = \mathbb{C}aP^2$ or $G_2/SO(4)$ [17]-[19], $M = SO(n+1)/SO(2) \times SO(n)$ or $M = Sp(n+1)/Sp(1) \times Sp(n)$ [22];

3. Δ_M acting on $\mathcal{S}^2(M, \mathbb{C})$ and M is the complex projective space $P^2(\mathbb{C})$ [23];

4. Δ_M acting on $\mathcal{S}^2(M, \mathbb{C})$ and M is either S^n or $P^n(\mathbb{C})$ [6] and [7];

5. Brian and Richard Millman gave in [2] a theoretical method for computing the spectra of Lichnerowicz Laplacian acting on $\Omega(G)$ where G is a compact semisimple Lie group endowed with the biinvariant metric induced from the negative of the Killing form;

6. Some partial results where given in [13]-[15];

7. Δ_M acting on $\mathcal{S}(M, \mathbb{C})$ and M is S^n [8].

In this paper, we compute the eigenvalues and we determine the spaces of eigentensors with its multiplicities of Δ_M acting on $\mathcal{S}(M, \mathbb{C})$ in the case where M is the complex projective space $P^n(\mathbb{C})$ endowed with the Kählerian structure quotient of the canonical Kählerian structure of $\mathbb{C}^{n+1}$. We use a fairly simple method which requires, in places, massive computations. Let us describe this method briefly.
First, since there is a natural map

$$\phi : \Gamma_Z(\otimes^* T^* \mathbb{C}^{n+1}, \mathbb{C}) \longrightarrow \Gamma(\otimes^* T^* P^n(\mathbb{C}), \mathbb{C}),$$

where $\Gamma_Z(\otimes^* T^* \mathbb{C}^{n+1}, \mathbb{C})$ is the space of complex covariant tensor fields invariant by the natural action of the circle on $\mathbb{C}^{n+1}$, we compute

$$\phi \circ \Delta_{\mathbb{C}^{n+1}} - \Delta_{P^n(\mathbb{C})} \circ \phi.$$

The formula obtained (cf. Theorem 2.1) involves natural operators on $\mathbb{C}^{n+1}$ and constitutes the principal tool of this paper. Hereafter, in Section 3, we adapt to our situation the methods used in [11] in the context of differential forms. Indeed, we consider, for any $p, q, k, l \in \mathbb{N}$, the space $T^{p,q}_{k,l}$ of traceless symmetric tensor field T on $\mathbb{C}^{n+1}$ of the form

$$T = \sum_{\substack{0\leq i_1<\ldots,<i_p\leq n \\ 0\leq j_1<\ldots,<j_q\leq n}} T_{i_1,\ldots,i_p,j_1,\ldots,j_q} dz_{i_1} \odot \ldots \odot dz_{i_p} \odot d\bar{z}_{j_1} \odot \ldots \odot d\bar{z}_{j_q},$$

where $T_{i_1,\ldots,i_p,j_1,\ldots,j_q}$ are harmonic polynomials of degree k with respect $z_0, \ldots, z_n$ and of degree l with respect $\bar{z}_0, \ldots, \bar{z}_n$ and such that the divergence of T vanishes. We have, if $<\,,\,>$ denotes the Euclidian metric on $\mathbb{C}^{n+1}$,

$$\phi : \bigoplus_{\substack{0\leq m\leq \min(p,q) \\ k+p=l+q}} <\,,\,>^m \odot T^{p-m,q-m}_{k,l} \longrightarrow \mathcal{S}^{p,q}(P^n(\mathbb{C}), \mathbb{C})$$

is injective and its image is dense. By introducing an algebraic lemma (cf. Lemma 3.3) we get a direct sum decomposition of any $T^{p,q}_{k,l}$, and we use the formula obtained in Theorem 2.1 to show that the image by ϕ of the spaces composing this direct sum are, actually, eigenspaces of $\Delta_{P^n(\mathbb{C})}$. We compute the multiplicities of these eigenspaces and we get the result desired (see Theorems 3.2 and 3.3). Finally,, we tabulate the results for the low values of p and q (Tables I-VIII) and, in particular, we recover the results obtained in [23] (Tables VI-VIII).

2 A relation between $\Delta_{\mathbb{C}^{n+1}}$ and $\Delta_{P^n(\mathbb{C})}$

The main result of this section is Theorem 2.1 which establishes a formula relating the Lichnerowicz Laplacian on $\mathbb{C}^{n+1}$ and the Lichnerowicz Laplacian on $P^n(\mathbb{C})$. This formula is the principal tool of this paper and its statement requires the introduction of some definitions and notations. Also we need to recall some basic properties of the Lichnerowicz Laplacian and to collect the basic material which will be used throughout the paper.

Let (M, g) be a Riemannian n-manifold. The curvature tensor field R of the Levi-Civita connection D associated to g is given by

$$R(X,Y)Z = D_{[X,Y]}Z - (D_X D_Y Z - D_Y D_X Z)\,,$$

and its Ricci endomorphism field $r : TM \longrightarrow TM$ is given by

$$g(r(X), Y) = \sum_{i=1}^{n} g(R(X, E_i)Y, E_i),$$

where $(E_1, \ldots, E_n)$ is any local orthonormal frame.
For any $p \in \mathbb{N}$, the connection D induces a differential operator

$$D : \Gamma(\otimes^p T^*M, \mathbb{C}) \longrightarrow \Gamma(\otimes^{p+1} T^*M, \mathbb{C})$$

given by

$$DT(X, Y_1, \ldots, Y_p) := D_X T(Y_1, \ldots, Y_p) = X.T(Y_1, \ldots, Y_p) - \sum_{j=1}^{p} T(Y_1, \ldots, D_X Y_j, \ldots, Y_p).$$

Its formal adjoint $D^* : \Gamma(\otimes^{p+1} T^*M, \mathbb{C}) \longrightarrow \Gamma(\otimes^p T^*M, \mathbb{C})$ is given by

$$D^*T(Y_1, \ldots, Y_p) = -\sum_{j=1}^{n} D_{E_i} T(E_i, Y_1, \ldots, Y_p),$$

where $(E_1, \ldots, E_n)$ is any local orthonormal frame.
We denote by δ the restriction of D^* to $\mathcal{S}(M, \mathbb{C})$ and we define $\delta^* : \mathcal{S}^p(M, \mathbb{C}) \longrightarrow \mathcal{S}^{p+1}(M, \mathbb{C})$ by

$$\delta^* T(X_1, \ldots, X_{p+1}) = \sum_{j=1}^{p+1} D_{X_j} T(X_1, \ldots, \hat{X}_j, \ldots, X_{p+1}),$$

where the symbol $\hat{}$ means that the term is omitted.
Recall that the operator trace $\mathrm{Tr}: \mathcal{S}^p(M,\mathbb{C}) \longrightarrow \mathcal{S}^{p-2}(M,\mathbb{C})$ is given by

$$\mathrm{Tr}T(X_1,\ldots,X_{p-2}) = \sum_{j=1}^{n} T(E_j,E_j,X_1,\ldots,X_{p-2}),$$

where $(E_1,\ldots,E_n)$ is any local orthonormal frame.
The Lichnerowicz Laplacian is the second order differential operator

$$\Delta_M : \Gamma(\otimes^p T^*M,\mathbb{C}) \longrightarrow \Gamma(\otimes^p T^*M,\mathbb{C})$$

given by

$$\Delta_M(T) = D^*D(T) + R(T),$$

where $R(T)$ is the curvature operator given by

$$R(T)(X_1,\ldots,X_p) = \sum_{j=1}^{p} T(X_1,\ldots,r(X_j),\ldots,X_p)$$
$$- \sum_{i<j}\sum_{l=1}^{n} \{T(X_1,\ldots,E_l,\ldots,R(X_i,E_l)X_j,\ldots,X_p) + T(X_1,\ldots,R(X_j,E_l)X_i,\ldots,E_l,\ldots,X_p)\},$$

where $(E_1,\ldots,E_n)$ is any local orthonormal frame and, in

$$T(X_1,\ldots,E_l,\ldots,R(X_i,E_l)X_j,\ldots,X_p),$$

E_l takes the place of X_i and $R(X_i,E_l)X_j$ takes the place of X_j.
This differential operator, introduced by Lichnerowicz in [16] pp. 26, is self-adjoint, elliptic and respects the symmetries of tensor fields. In particular, Δ_M leaves invariant $\mathcal{S}(M,\mathbb{C})$ and the restriction of Δ_M to $\Omega(M,\mathbb{C})$ coincides with the Hodge-de Rham Laplacian.
Note that if $T \in \mathcal{S}(M)$ then

$$\mathrm{Tr}(\Delta_M T) = \Delta_M(\mathrm{Tr}T), \qquad (1)$$
$$\Delta_M(T \odot g) = (\Delta_M T) \odot g, \qquad (2)$$

where $\odot$ is the symmetric product.
The Lichnerowicz Laplacian is compatible with Kählerian structures. Indeed, suppose that (M,g) is Kählerian, i.e., there exists a complex structure J on

M such that $DJ = 0$ and g is Hermitian with respect to J. The complex structure J defines a bigraduation

$$\mathcal{S}^p(M, \mathbb{C}) = \bigoplus_{r+q=p} \mathcal{S}^{r,q}(M),$$

and Δ_M respects this bigraduation.
For any $T \in \Gamma(\otimes^p T^*M, \mathbb{C})$, for any vector field Y and for any $1 \leq i < j \leq p$, we denote by $i_{Y,j}T$ the $(p-1)$-tensor field given by

$$i_{Y,j}T(X_1, \dots, X_{p-1}) = T(X_1, \dots, X_{j-1}, Y, X_j, \dots, X_{p-1}),$$

by $\mathrm{Tr}_{i,j}T$ the $(p-2)$-tensor field given by

$$\mathrm{Tr}_{i,j}T(X_1, \dots, X_{p-2}) = \sum_{l=1}^{n} T(X_1, \dots, X_{i-1}, E_l, X_i, \dots, X_{j-2}, E_l, X_{j-1}, \dots, X_{p-2}),$$

and by $\mathrm{Tr}_{i,j,J}T$ the $(p-2)$-tensor field given by

$$\mathrm{Tr}_{i,j,J}T(X_1, \dots, X_{p-2}) = \sum_{l=1}^{n} T(X_1, \dots, X_{i-1}, E_l, X_i, \dots, X_{j-2}, JE_l, X_{j-1}, \dots, X_{p-2}),$$

where $(E_1, \dots, E_n)$ is any local orthonormal frame of M.

Remark 2.1 *If T is a complex symmetric covariant tensor field then*

$$\mathrm{Tr}_{i,j}T = \mathrm{Tr}T \quad \textit{and} \quad \mathrm{Tr}_{i,j,J}T = 0.$$

For any permutation σ of $\{1, \dots, p\}$, we denote by T^σ the p-tensor field

$$T^\sigma(X_1, \dots, X_p) = T(X_{\sigma(1)}, \dots, X_{\sigma(p)}).$$

For $1 \leq i < j \leq p$, the transposition of (i, j) is the permutation $\sigma_{i,j}$ of $\{1, \dots, p\}$ such that $\sigma_{i,j}(i) = j$, $\sigma_{i,j}(j) = i$ and $\sigma_{i,j}(k) = k$ for $k \neq i, j$. We shall denote by $\mathcal{T}$ the set of the transpositions of $\{1, \dots, p\}$.

On other hand, for $p \geq 2$, we denote by T^J and $\overline{T}^J$ the p-tensors fields given by

$$T^J(X_1, \dots, X_p) = \sum_{i<j} T(X_1, \dots, JX_i, \dots, JX_j, \dots, X_p),$$
$$\overline{T}^J(X_1, \dots, X_p) = \sum_{i<j} T(X_1, \dots, JX_j, \dots, JX_i, \dots, X_p).$$

Finally, we define $\delta^{*c} : \mathcal{S}^p(M,\mathbb{C}) \longrightarrow \mathcal{S}^{p+1}(M,\mathbb{C})$ by

$$\delta^{*c}T(X_1,\ldots,X_{p+1}) = \sum_{j=1}^{p+1} D_{JX_j}T(X_1,\ldots,\hat{X}_j,\ldots,X_{p+1}),$$

and we put

$$\delta_h^* = \frac{1}{2}(\delta^* - i\delta^{*c}) \quad \text{and} \quad \overline{\delta_h^*} = \frac{1}{2}(\delta^* + i\delta^{*c}).$$

Note that

$$\delta_h^* \circ \overline{\delta_h^*} = \overline{\delta_h^*} \circ \delta_h^*. \tag{3}$$

The complex projective space $P^n(\mathbb{C})$ inherits a natural Kählerian structure from $\mathbb{C}^{n+1}$, let us describe this structure and introduce some notations.
Let $(z_0,\ldots,z_n)$ be the standard holomorphic coordinates on $\mathbb{C}^{n+1}$. Put $z_i = x_i + \sqrt{-1}y_i$,

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right).$$

The standard complex structure J_0 of $\mathbb{C}^{n+1}$ is given by

$$J_0\frac{\partial}{\partial z_i} = \sqrt{-1}\frac{\partial}{\partial z_i} \quad \text{and} \quad J_0\frac{\partial}{\partial \bar{z}_i} = -\sqrt{-1}\frac{\partial}{\partial \bar{z}_i}.$$

Let $< \, , \, >= \sum_{i=0}^n dz_i.d\bar{z}_i$ be the flat Kähler metric on $\mathbb{C}^{n+1}$ and let $\Omega_0 = -\sqrt{-1}\sum_{i=0}^n dz_i \wedge d\bar{z}_i$ be its Kähler form.
The radial vector field $\overrightarrow{r} = \sum_{i=0}^n \left(x_i\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial y_i}\right)$ splits to $\overrightarrow{r} = W + \overline{W}$, where

$$W = \sum_{i=0}^n z_i\frac{\partial}{\partial z_i} \quad \text{and} \quad \overline{W} = \sum_{i=0}^n \bar{z}_i\frac{\partial}{\partial \bar{z}_i}.$$

Put $Z = J_0\overrightarrow{r}$.
The differential of $r^2 = \sum_{i=0}^n |z_i|^2$ splits to $dr^2 = W^* + \overline{W}^*$, where

$$W_0^* = \sum_{i=0}^n \bar{z}_i dz_i, \quad \text{and} \quad \overline{W}_0^* = \sum_{i=0}^n z_i d\bar{z}_i.$$

Let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow P^n(\mathbb{C})$ be the natural projection and $\pi_s : S^{2n+1} \longrightarrow P^n(\mathbb{C})$ its restriction to $S^{2n+1} \subset \mathbb{C}^{n+1} \setminus \{0\}$. For any $m \in S^{2n+1}$, put $F_m = \ker((\pi_s)_*)_m$ and let $F_m^{\perp}$ be the orthogonal complementary subspace to F_m in $T_m(S^{2n+1})$;

$$T_m(S^{2n+1}) = F_m \oplus F_m^{\perp}.$$

We introduce the Riemannian metric g on $P^n(\mathbb{C})$ so that the restriction of $(\pi_s)_*$ to $F_m^{\perp}$ is an isometry onto $T_{\pi(m)}(P^n(\mathbb{C}))$. The standard complex structures J on $P^n(\mathbb{C})$ is given by

$$J(\pi_s)_* v = (\pi_s)_* J_0 v,\ v \in F_m^{\perp}.$$

For any vector field X tangent to $P^n(\mathbb{C})$, there exists an unique vector field X^h tangent to S^{2n+1} satisfying, for any $m \in S^{2n+1}$,

$$X^h(m) \in F_m^{\perp} \quad \text{and} \quad (\pi_s)_*(X^h) = X.$$

The vector field X^h is the horizontal left of X.
For any $p, q \in \mathbb{N}$, we denote by $\Gamma_Z(\otimes^p T^* \mathbb{C}^{n+1}, \mathbb{C})$ and $\mathcal{S}_Z^{p,q}(\mathbb{C}^{n+1})$ the space of complex covariant p-tensor fields on $\mathbb{C}^{n+1}$ and the space of complex symmetric covariant tensor fields of type (p,q) on $\mathbb{C}^{n+1}$, respectively, which are invariant by Z. A tensor field T belongs to $\Gamma_Z(\otimes^p T^* \mathbb{C}^{n+1})$ if and only if $L_Z T = 0$.
We define a linear map

$$\phi : \Gamma_Z(\otimes^p T^* \mathbb{C}^{n+1}, \mathbb{C}) \longrightarrow \Gamma(\otimes^p T^* P^n(\mathbb{C}), \mathbb{C})$$

by

$$\phi(T)(X_1, \ldots, X_p)(\pi_s(m)) = T(X_1^h, \ldots, X_p^h)(m), \quad m \in S^{2n+1}.$$

The map ϕ is well defined and $\phi\,(\mathcal{S}_Z^{p,q}(\mathbb{C}^{n+1})) \subset \mathcal{S}^{p,q}(P^n(\mathbb{C}), \mathbb{C})$.
Note that the Kähler form Ω of $P^n(\mathbb{C})$ satisfies $\Omega = \phi(\Omega_0)$.

The Lichnerowicz Laplacian on $P^n(\mathbb{C})$ involves the curvature operator and we will compute it now.

Lemma 2.1 *The tensor curvature R and the Ricci endomorphism field r associated to the Riemannian metric g on $P^n(\mathbb{C})$ are given by*

$$\begin{aligned} R(X_1, X_2)X_3 &= g(X_1, X_3)X_2 - g(X_2, X_3)X_1 - 2g(JX_2, X_1)JX_3 + g(JX_3, X_2)JX_1 \\ &\quad -g(JX_3, X_1)JX_2 \\ r(X) &= 2(n+1)X. \end{aligned}$$

Proof. These formulas can be deduced easily from the curvature of S^{2n+1} by using the Riemannian submersion $\pi_s : S^{2n+1} \longrightarrow P^n(\mathbb{C})$ and the O'Neil formulas (see for instance [5, pp. 241]).$\square$

A direct computation using Lemma 2.1 and the definition of the curvature operator gives the following lemma.

Lemma 2.2 *For any covariant p-tensor field T on $P^n(\mathbb{C})$, we have*

$$\begin{aligned} R(T)(X_1, \ldots, X_p) &= 2(n+1)pT(X_1, \ldots, X_p) - 4T^J(X_1, \ldots, X_p) \\ &\quad -2\overline{T}^J(X_1, \ldots, X_p) + 2\sum_{\sigma \in \mathcal{T}} T^\sigma(X_1, \ldots, X_p) \\ &\quad -2\sum_{i<j} g(X_i, X_j)\mathrm{Tr}_{i,j}T(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p) \\ &\quad -2\sum_{i<j} \Omega(X_i, X_j)\mathrm{Tr}_{i,j,J}T(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p). \end{aligned}$$

Now we are able to state the main result of this section.

Theorem 2.1 *Let $T \in \Gamma_Z(\otimes^p T^*\mathbb{C}^{n+1}, \mathbb{C})$. Then*

$$\begin{aligned} \phi(\Delta_{\mathbb{C}^{n+1}}T) - \Delta_{P^n(\mathbb{C})}\phi(T) &= \phi\Big(p(1-p)T + 2(p-n)L_{\overrightarrow{r}}T - L_{\overrightarrow{r}} \circ L_{\overrightarrow{r}}T \\ &\quad -2\sum_{\sigma \in \mathcal{T}} T^\sigma + 2T^{J_0} + 2\overline{T}^{J_0} + O(T)\Big), \end{aligned}$$

where

$$\begin{aligned} &O(T)(X_1, \ldots, X_p) = \\ &+2\sum_{j=1}^{p}\Big(D_{J_0X_j}(i_{J_0\overrightarrow{r},j}T)(X_1, \ldots, \hat{X}_j, \ldots, X_p) - D_{X_j}(i_{\overrightarrow{r},j}T)(X_1, \ldots, \hat{X}_j, \ldots, X_p)\Big) \\ &+2\sum_{i<j} < X_i, X_j > \mathrm{Tr}_{i,j}T(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p) \\ &+2\sum_{i<j} \Omega_0(X_i, X_j)\mathrm{Tr}_{i,j,J_0}T(X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p). \end{aligned}$$

Proof. The proof is a massive computation in a local orthonormal frame. For any vector field X tangent to $P^n(\mathbb{C})$, even if its horizontal left X^h is a vector field tangent to S^{2n+1}, sometimes we need to extent it to a local vector field on $\mathbb{C}^{n+1}$ and we continue to note it by X^h.

We choose a local orthonormal frame of $\mathbb{C}^{n+1}$ of the form $(E_1^h, \ldots, E_{2n}^h, N, J_0 N)$ in a neighborhood of a point $m \in S^{2n+1}$ such that $(E_1^h, \ldots, E_{2n}^h)$ is the horizontal left of a local orthonormal frame $(E_1, \ldots, E_{2n})$ of $P^n(\mathbb{C})$ in a neighborhood of $\pi_s(m)$ and $N = \frac{1}{r}\overrightarrow{r}$ where $r = \sqrt{|z_0|^2 + \ldots + |z_n|^2}$.
Let D be the Levi-Civita connection associated to the flat Riemannian metric on $\mathbb{C}^{n+1}$. For any vector field X on $\mathbb{C}^{n+1}$, we have

$$\begin{aligned}
D_X N &= \frac{1}{r}\left(X - < X, N > N\right), && (4)\\
D_N X &= [N, X] + \frac{1}{r}(X - < X, N > N), && (5)\\
D_X J_0 N &= \frac{1}{r}(J_0 X - < X, N > J_0 N), && (6)\\
D_{J_0 N} X &= [J_0 N, X] + \frac{1}{r}(J_0 X - < X, N > J_0 N). && (7)
\end{aligned}$$

Let ∇ be the Levi-Civita connexion of the Riemannian metric g on $P^n(\mathbb{C})$. We have, for any vector fields X, Y tangent to $P^n(\mathbb{C})$ and in restriction to S^{2n+1},

$$D_{X^h} Y^h = (\nabla_X Y)^h + \frac{1}{r} < J_0 Y^h, X^h > J_0 N - \frac{1}{r} < X^h, Y^h > N. \qquad (8)$$

Let T be a covariant p-tensor field on $\mathbb{C}^{n+1}$ such that $L_Z T = 0$ and $(X_1, \ldots, X_p)$ a family of vector fields on $P^n(\mathbb{C})$ in a neighborhood of $\pi_s(m)$.
A direct calculation using the definition of the Lichnerowicz Laplacian gives

$$\Delta_{\mathbb{C}^{n+1}}(T)(X_1^h, \ldots, X_p^h) = D^* D(T)(X_1^h, \ldots, X_p^h) = Q_1 + Q_2 + Q_3 + Q_4,$$

where

$$\begin{aligned}
Q_1 = \sum_{i=1}^{2n} \Bigg(& -E_i^h E_i^h . T(X_1^h, \ldots, X_p^h) + 2 \sum_{j=1}^{p} E_i^h . T(X_1^h, \ldots, D_{E_i^h} X_j^h, \ldots, X_p^h) \\
& + D_{E_i^h} E_i^h . T(X_1^h, \ldots, X_p^h) - \sum_{j=1}^{p} T(X_1^h, \ldots, D_{D_{E_i^h} E_i^h} X_j^h, \ldots, X_p^h) \\
& - \sum_{j=1}^{p} T(X_1^h, \ldots, D_{E_i^h} D_{E_i^h} X_j^h, \ldots, X_p^h) \Bigg),
\end{aligned}$$

$$
\begin{aligned}
Q_2 &= -2\sum_{i=1}^{2n}\sum_{l<j} T(X_1^h,\ldots,D_{E_i^h}X_l^h,\ldots,D_{E_i^h}X_j^h,\ldots,X_p^h),\\
Q_3 &= -N.N.T(X_1^h,\ldots,X_p^h)+2\sum_{j=1}^{p} N.T(X_1^h,\ldots,D_N X_j^h,\ldots,X_p^h)\\
&\quad +D_N N.T(X_1^h,\ldots,X_p^h)-\sum_{j=1}^{p} T(X_1^h,\ldots,D_{D_N N}X_j^h,\ldots,X_p^h)\\
&\quad -\sum_{j=1}^{p} T(X_1^h,\ldots,D_N D_N X_j^h,\ldots,X_p^h)-2\sum_{l<j} T(X_1^h,\ldots,D_N X_l^h,\ldots,D_N X_j^h,\ldots,X_p^h),\\
Q_4 &= -J_0N.J_0N.T(X_1^h,\ldots,X_p^h)+2\sum_{j=1}^{p} J_0N.T(X_1^h,\ldots,D_{J_0N}X_j^h,\ldots,X_p^h)\\
&\quad +D_{J_0N}J_0N.T(X_1^h,\ldots,X_p^h)-\sum_{j=1}^{p} T(X_1^h,\ldots,D_{D_{J_0N}J_0N}X_j^h,\ldots,X_p^h)\\
&\quad -\sum_{j=1}^{p} T(X_1^h,\ldots,D_{J_0N}D_{J_0N}X_j^h,\ldots,X_p^h)\\
&\quad -2\sum_{l<j} T(X_1^h,\ldots,D_{J_0N}X_l^h,\ldots,D_{J_0N}X_j^h,\ldots,X_p^h).
\end{aligned}
$$

By using $(4)-(8)$, we get

$$
\begin{aligned}
D_{D_{E_i^h}E_i^h}X_j^h &= (\nabla_{\nabla_{E_i}E_i}X_j)^h+<(\nabla_{E_i}E_i)^h,(JX_j)^h>J_0N-<(\nabla_{E_i}E_i)^h,X_j^h>N\\
&\quad -[N,X_j]-X_j, &(9)\\
D_{E_i}D_{E_i}X_j &= (\nabla_{E_i}\nabla_{E_i}X_j)^h+<E_i^h,(\nabla_{E_i}JX_j)^h>J_0N-<E_i^h,(\nabla_{E_i}X_j)^h>N\\
&\quad +E_i^h.<E_i^h,(JX_j)^h>J_0N-E_i^h.<E_i^h,X_j^h>N\\
&\quad +<E_i^h,(JX_j)^h>J_0E_i^h-<E_i^h,X_j^h>E_i^h, &(10)\\
D_ND_NX_j^h &= [N,[N,X_j^h]]+\frac{2}{r}[N,X_j^h]+(\frac{1}{r^2}-\frac{1}{r})(X_j^h-<X_j^h,N>N)\\
&\quad -\frac{2}{r}N.<X_j^h,N>N, &(11)\\
D_{J_0N}D_{J_0N}X_j^h &= [J_0N,[J_0N,X_j^h]]+\frac{2}{r}[J_0N,J_0X_j^h]-\frac{1}{r^2}X_j^h\\
&\quad -\frac{2}{r}<D_{J_0N}X_j^h,N>J_0N. &(12)
\end{aligned}
$$

A careful verification using (8) − (10) leads to

$$\begin{aligned}
Q_1 &= \sum_{i=1}^{2n}\Bigg(-E_iE_i.\phi(T)(X_1,\ldots,X_p)+2\sum_{j=1}^{p}E_i.\phi(T)(X_1,\ldots,\nabla_{E_i}X_j,\ldots,X_p)\\
&\quad+\nabla_{E_i}E_i.\phi(T)(X_1,\ldots,X_p)-\sum_{j=1}^{p}\phi(T)(X_1,\ldots,\nabla_{\nabla_{E_i}E_i}X_j,\ldots,X_p)\\
&\quad-\sum_{j=1}^{p}\phi(T)(X_1,\ldots,\nabla_{E_i}\nabla_{E_i}X_j,\ldots,X_p)\Bigg)\\
&\quad+2p(n+1)\phi(T)(X_1,\ldots,X_p)-2nL_NT(X_1^h,\ldots,X_p^h)\\
&\quad+2\sum_{j=1}^{p}\Bigg(J_0X_j^h.T(X_1^h,\ldots,\overbrace{J_0N}^{j},\ldots,X_p^h)-X_j^h.T(X_1^h,\ldots,\overbrace{N}^{j},\ldots,X_p^h)\Bigg).
\end{aligned}$$

Now, by using (8), we get

$$\begin{aligned}
Q_2 &= -2\sum_{l<j}\sum_{i=1}^{2n}\phi(T)(X_1,\ldots,\nabla_{E_i}X_l,\ldots,\nabla_{E_i}X_j,\ldots,X_p)\\
&+2\sum_{l<j}T(X_1^h,\ldots,\overbrace{N}^{l},\ldots,D_{X_l^h}X_j^h,\ldots,X_p^h)+2\sum_{l<j}T(X_1^h,\ldots,D_{X_j^h}X_l^h,\ldots,\overbrace{N}^{j},\ldots,X_p^h)\\
&-2\sum_{l<j}T(X_1^h,\ldots,\overbrace{J_0N}^{l},\ldots,D_{J_0X_l^h}X_j^h,\ldots,X_p^h)-2\sum_{l<j}T(X_1^h,\ldots,D_{J_0X_j^h}X_l^h,\ldots,\overbrace{J_0N}^{j},\ldots,X_p^h)\\
&+2\sum_{l<j}<X_l^h,X_j^h>\Bigg(T(X_1^h,\ldots,\overbrace{N}^{l},\ldots,\overbrace{N}^{j},\ldots,X_p^h)+T(X_1^h,\ldots,\overbrace{J_0N}^{l},\ldots,\overbrace{J_0N}^{j},\ldots,X_p^h)\Bigg)\\
&-2\sum_{l<j}<J_0X_l^h,X_j^h>\Bigg(T(X_1^h,\ldots,\overbrace{J_0N}^{l},\ldots,\overbrace{N}^{j},\ldots,X_p^h)-T(X_1^h,\ldots,\overbrace{N}^{l},\ldots,\overbrace{J_0N}^{j},\ldots,X_p^h)\Bigg).
\end{aligned}$$

We deduce that

$$\begin{aligned}
Q_1+Q_2 &= \nabla^*\nabla\phi(T)(X_1,\ldots,X_p)+2p(n+1)\phi(T)(X_1,\ldots,X_p)-2nL_NT(X_1^h,\ldots,X_p^h)\\
&\quad+2\sum_{j=1}^{p}\left(D_{J_0X_j^h}(i_{J_0N,j}T)(X_1^h,\ldots,\hat{X}_j^h,\ldots,X_p^h)-D_{X_j^h}(i_{N,j}T)(X_1^h,\ldots,\hat{X}_j^h,\ldots,X_p^h)\right)
\end{aligned}$$

$$+2\sum_{l<j} < X_l^h, X_j^h > \left(T(X_1^h, \dots, \overbrace{N}^{l}, \dots, \overbrace{N}^{j}, \dots, X_p^h) + T(X_1^h, \dots, \overbrace{J_0N}^{l}, \dots, \overbrace{J_0N}^{j}, \dots, X_p^h) \right)$$

$$+2\sum_{l<j} \Omega_0(X_l^h, X_j^h) \left(T(X_1^h, \dots, \overbrace{N}^{l}, \dots, \overbrace{J_0N}^{j}, \dots, X_p^h) + T(X_1^h, \dots, \overbrace{J_0N}^{l}, \dots, \overbrace{J_0J_0N}^{j}, \dots, X_p^h) \right).$$

Remark that $\Delta_{P^n(\mathbb{C})}\phi(T) = \nabla^*\nabla\phi(T) + R(\phi(T))$ where $R(\phi(T))$ is given by Lemma 2.2. We deduce hence

$$\begin{aligned}
&Q_1 + Q_2 - \Delta_{P^n(\mathbb{C})}\phi(T)(X_1, \dots, X_p) = -2nL_N T(X_1^h, \dots, X_p^h) + 4T^{J_0}(X_1^h, \dots, X_p^h) \\
&+2\overline{T}^{J^0}(X_1^h, \dots, X_p^h) - 2\sum_{\sigma\in\mathcal{T}} T^\sigma(X_1^h, \dots, X_p^h) \\
&+2\sum_{j=1}^{p} \Big(D_{J_0X_j^h}(i_{J_0N,j}T)(X_1^h, \dots, \hat{X}_j^h, \dots, X_p^h) - D_{X_j^h}(i_{N,j}T)(X_1^h, \dots, \hat{X}_j^h, \dots, X_p^h) \Big) \\
&+2\sum_{i<j} < X_i^h, X_j^h > \mathrm{Tr}_{i,j}T(X_1^h, \dots, \hat{X}_i^h, \dots, \hat{X}_j^h, \dots, X_p^h) \\
&+2\sum_{i<j} \Omega_0(X_i^h, X_j^h)\mathrm{Tr}_{i,j,J_0}T(X_1^h, \dots, \hat{X}_i^h, \dots, \hat{X}_j^h, \dots, X_p^h). \qquad (13)
\end{aligned}$$

Let us compute Q_3. Now by using (5) and (11) and by taking the restriction to S^{2n+1}, we have

$$\begin{aligned}
2\sum_{j=1}^{p} N.T(X_1^h, \dots, D_N X_j^h, \dots, X_p^h) &= 2\sum_{j=1}^{p} N.T(X_1^h, \dots, [N, X_j^h], \dots, X_p^h) \\
&\quad +2\sum_{j=1}^{p} N(\frac{1}{r})T(X_1^h, \dots, X_j^h, \dots, X_p^h) \\
&\quad +2\sum_{j=1}^{p} N.T(X_1^h, \dots, X_j^h, \dots, X_p^h) \\
&\quad -2\sum_{j=1}^{p} N(< X_j^h, N >)T(X_1^h, \dots, \overbrace{N}^{j}, \dots, X_p^h) \\
&= 2\sum_{j=1}^{p} N.T(X_1^h, \dots, [N, X_j^h], \dots, X_p^h) \\
&\quad -2pT(X_1^h, \dots, X_p^h) + 2pN.T(X_1^h, \dots, X_p^h)
\end{aligned}$$

$$-2\sum_{j=1}^{p} N(< X_j^h, N >)T(X_1^h, \ldots, \overbrace{N}^{j}, \ldots, X_p^h).$$

$$\begin{aligned}
\sum_{j=1}^{p} T(X_1^h, \ldots, D_N D_N X_j^h, \ldots, X_p^h) \;&=\; 2\sum_{j=1}^{p} T(X_1^h, \ldots, [N, X_j^h], \ldots, X_p^h) \\
&\sum_{j=1}^{p} T(X_1^h, \ldots, [N, [N, X_j^h], \ldots, X_p^h) \\
&-2\sum_{j=1}^{p} N(< X_j^h, N >)T(X_1^h, \ldots, \overbrace{N}^{j}, \ldots, X_p^h).
\end{aligned}$$

$$\begin{aligned}
&\sum_{i<j} T(X_1^h, \ldots, D_N X_i^h, \ldots, D_N X_j^h, \ldots, X_p^h) = \\
&\sum_{i<j} T(X_1^h, \ldots, [N, X_i^h], \ldots, [N, X_j^h], \ldots, X_p^h) + \frac{p(p-1)}{2} T(X_1^h, \ldots, X_p^h) \\
&+\sum_{i<j} T(X_1^h, \ldots, X_i^h, \ldots, [N, X_j^h], \ldots, X_p^h) \\
&+\sum_{i<j} T(X_1^h, \ldots, [N, X_i^h], \ldots, X_j^h, \ldots, X_p^h) = \\
&\sum_{i<j} T(X_1^h, \ldots, [N, X_i^h], \ldots, [N, X_j^h], \ldots, X_p^h) + \frac{p(p-1)}{2} T(X_1^h, \ldots, X_p^h) \\
&+(p-1)\sum_{j=1}^{p} T(X_1^h, \ldots, [N, X_j^h], \ldots, X_p^h).
\end{aligned}$$

So we get, in restriction to S^{2n+1},

$$Q_3 = -L_N \circ L_N T(X_1^h, \ldots, X_p^h) + 2pL_N T(X_1^h, \ldots, X_p^h) - p(1+p)T(X_1^h, \ldots, X_p^h). \tag{14}$$

Let us compute Q_4. By using (7) et (12), we get in restriction to S^{2n+1},

$$\begin{aligned}
Q_4 \;=\; & -J_0N.J_0N.T(X_1^h, \ldots, X_p^h) + 2\sum_{j=1}^{p} J_0N.T(X_1^h, \ldots, [J_0N, X_j^h], \ldots, X_p^h) \\
& +2\sum_{j=1}^{p} J_0N.T(X_1^h, \ldots, J_0X_j^h, \ldots, X_p^h) \\
& -N.T(X_1^h, \ldots, X_p^h) + \sum_{j=1}^{p} T(X_1^h, \ldots, [N, X_j^h], \ldots, X_p^h) + pT(X_1^h, \ldots, X_p^h)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{p} T(X_1^h,\dots,[J_0N,[J_0N,X_j^h]],\dots,X_p^h) - 2\sum_{j=1}^{p} T(X_1^h,\dots,[J_0N,J_0X_j^h],\dots,X_p^h)\\
&+pT(X_1^h,\dots,X_p^h) - 2\sum_{i<j} T(X_1^h,\dots,[J_0N,X_i^h],\dots,[J_0N,X_j^h],\dots,X_p^h)\\
&-2\sum_{i<j} T(X_1^h,\dots,[J_0N,X_i^h],\dots,J_0X_j^h,\dots,X_p^h)\\
&-2\sum_{i<j} T(X_1^h,\dots,J_0X_i^h,\dots,[J_0N,X_j^h],\dots,X_p^h)\\
&-2\sum_{i<j} T(X_1^h,\dots,J_0X_i^h,\dots,J_0X_j^h,\dots,X_p^h).
\end{aligned}
$$

Hence

$$
\begin{aligned}
Q_4 \;=\; & -L_{J_0N}\circ L_{J_0N}T(X_1^h,\dots,X_p^h) - L_N T(X_1^h,\dots,X_p^h) + 2pT(X_1^h,\dots,X_p^h)\\
&+2\sum_{j=1}^{p} L_{J_0N}T(X_1^h,\dots,J_0X_j^h,\dots,X_p^h) - 2T^{J_0}(X_1^h,\dots,X_p^h). \qquad (15)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\phi(\Delta_{\mathbb{C}^{n+1}}T)(X_1,\dots,X_p) - \Delta_{P^n(\mathbb{C})}\phi(T)(X_1,\dots,X_p) \;=\; & Q_1+Q_2+Q_3+Q_4\\
& -\Delta_{P^n(\mathbb{C})}\phi(T)(X_1,\dots,X_p)
\end{aligned}
$$

and we get the desired formula by using (13)−(15), by noting that $L_{J_0N_0}T=0$ and by remarking that the following formulas holds in restriction to S^{2n+1}

$$
\begin{aligned}
\sum_{j=1}^{p} D_{J_0X_j^h}(i_{J_0N,j}T)(X_1^h,\dots,\hat{X}^h{}_j,\dots,X_p^h) \;&=\; \sum_{j=1}^{p} D_{J_0X_j^h}(i_{J_0\overrightarrow{r},j}T)(X_1^h,\dots,\hat{X}_j^h,\dots,X_p^h),\\
\sum_{j=1}^{p} D_{X_j^h}(i_{N,j}T)(X_1^h,\dots,\hat{X}_j^h,\dots,X_p^h) \;&=\; \sum_{j=1}^{p} D_{X_j^h}(i_{\overrightarrow{r},j}T)(X_1^h,\dots,\hat{X}_j^h,\dots,X_p^h),\\
L_N T = L_{\overrightarrow{r}}T \qquad \text{and} \quad & L_N\circ L_N T = -L_{\overrightarrow{r}}T + L_{\overrightarrow{r}}\circ L_{\overrightarrow{r}}T.
\end{aligned}
$$

□

3 Spectra and symmetric eigentensors of the Lichnerowicz Laplacian on $P^n(\mathbb{C})$

In this section, we formulate Theorem 2.1 in the context of symmetric covariant tensor fields (Theorem 3.1), we adapt to our context the results obtained

in [11] in the context of differential forms and we introduce an algebraic lemma (Lemma 3.3). Hereafter, we deduce from this lemma and Theorem 3.1 the mains results of this paper, namely, Theorems 3.2 and 3.3. Finally, we tabulate the eigenvalues and the eigenspaces of $\Delta_{P^n(\mathbb{C})}$ acting on differential 1-forms and on symmetric covariant 2-tensor fields (Tables II-VIII).

The following result is an immediate consequence of Theorem 2.1, Remark 2.1 and the definitions of W, $\overline{W}$, δ_h^* and $\overline{\delta_h^*}$.

Theorem 3.1 *Let T be a symmetric p-tensor field on $\mathbb{C}^{n+1}$ such that $L_Z T = 0$. Then*

$$\begin{aligned}\phi(\Delta_{\mathbb{C}^{n+1}}T) - \Delta_{P^n(\mathbb{C})}\phi(T) &= \phi\Big(2p(1-p)T + 2(p-n)L_{\overrightarrow{r}}T - L_{\overrightarrow{r}} \circ L_{\overrightarrow{r}}T \\ &\quad +4T^{J_0} - 4\overline{\delta_h^*}i_{\overline{W}}T - 4\delta_h^* i_W T + 2 <\,,\,> \odot \mathrm{Tr}T\Big).\end{aligned}$$

The results on harmonic homogeneous forms on $\mathbb{C}^{n+1}$, obtained by Ikeda and Taniguchi in [11], can be adapted easily to get similar results on harmonic homogeneous symmetric covariant tensor fields.
Let $\mathcal{S}P_{k,l}^{p,q}$ be the set of symmetric tensor field T on $\mathbb{C}^{n+1}$ of the form

$$T = \sum_{\substack{0\le i_1<\ldots,<i_p\le n \\ 0\le j_1<\ldots,<j_q\le n}} T_{i_1,\ldots,i_p,j_1,\ldots,j_q} dz_{i_1}\odot\ldots\odot dz_{i_p}\odot d\bar{z}_{j_1}\odot\ldots\odot d\bar{z}_{j_q},$$

where $T_{i_1,\ldots,i_p,j_1,\ldots,j_q}$ are polynomials of degree k with respect $z_0,\ldots,z_n$ and of degree l with respect $\bar{z}_0,\ldots,\bar{z}_n$. Put

$$\mathcal{S}H_{k,l}^{p,q} = \mathcal{S}P_{k,l}^{p,q} \cap \ker\delta \cap \ker\Delta_{\mathbb{C}^{n+1}} \quad \text{and} \quad T_{k,l}^{p,q} = \mathcal{S}H_{k,l}^{p,q} \cap \ker\mathrm{Tr}.$$

In the same way as [11, Lemma 6.4], we have

$$\mathcal{S}P_{k,l}^{p,q} = \mathcal{S}H_{k,l}^{p,q} \oplus \Big(W^* \odot \mathcal{S}P_{k,l-1}^{p-1,q} + \overline{W}^* \odot \mathcal{S}P_{k-1,l}^{p,q-1} + r^2\mathcal{S}P_{k-1,l-1}^{p,q}\Big). \qquad (16)$$

Furthermore, in the same way as [11, Corollary 7.11], we get the following Lemma.

Lemma 3.1 $\phi : \bigoplus_{\substack{0\le m\le\min(p,q) \\ k+p=l+q}} <\,,\,>^m \odot T_{k,l}^{p-m,q-m} \longrightarrow \mathcal{S}^{p,q}(P^n(\mathbb{C}),\mathbb{C})$ *is injective and its image is dense.*

The following lemma can be obtained easily by a direct computation.

Lemma 3.2 *For $T \in \mathcal{S}P^{p,q}_{k,l}$, we have*

1. $i_W \delta^*_h T - \delta^*_h i_W T = (k-p)T$;
2. $i_{\overline{W}} \overline{\delta^*_h} T - \overline{\delta^*_h} i_{\overline{W}} T = (l-q)T$;
3. $i_{\overline{W}} \delta^*_h T - \delta^*_h i_{\overline{W}} T = 0$;
4. $i_W \overline{\delta^*_h} T - \overline{\delta^*_h} i_W T = 0$.

Note that the operators $i_W, i_{\overline{W}}, \delta^*_h$ and $\overline{\delta^*_h}$ preserve the spaces $T^{p,q}_{k,l}$, namely,

$$\begin{array}{rcl} i_W : T^{p,q}_{k,l} \longrightarrow T^{p-1,q}_{k+1,l} & , & i_{\overline{W}} : T^{p,q}_{k,l} \longrightarrow T^{p,q-1}_{k,l+1}, \\ \delta^*_h : T^{p,q}_{k,l} \longrightarrow T^{p+1,q}_{k-1,l} & , & \overline{\delta^*_h} : T^{p,q}_{k,l} \longrightarrow T^{p,q+1}_{k,l-1}. \end{array} \tag{17}$$

The task is now to decompose $T^{p,q}_{k,l}$ as a direct sum of spaces whose the images by ϕ are eigenspaces of $\Delta_{P^n(\mathbb{C})}$ and get, according to Lemma 3.1 and (2), all the eigenvalues. This decomposition is based on the following algebraic lemma.

Lemma 3.3 *Let V be a finite dimensional vectorial space, ϕ and ψ are two endomorphisms of V and $(A^p_k)_{k,p\in\mathbb{N}\cup\{-1\}}$ is a family of vectorial subspaces of V such that:*

1. *for any $p, k \in \mathbb{N}$, $A^p_{-1} = A^{-1}_k = \{0\}$;*
2. *for any $p, k \in \mathbb{N}$, $\phi(A^p_k) \subset A^{p+1}_{k-1}$ and $\psi(A^p_k) \subset A^{p-1}_{k+1}$;*
3. *for any $p, k \in \mathbb{N}$ and for any $a \in A^p_k$,*
$$\phi \circ \psi(a) - \psi \circ \phi(a) = (p-k)a.$$

Then:
(i) for any $k < p$, $\psi : A^p_k \longrightarrow A^{p-1}_{k+1}$ is injective;
(ii) for any $k \le p$, we have

$$A^p_k = (A^p_k \cap \ker\phi) \oplus \psi(A^{p+1}_{k-1}) \quad \text{and} \quad A^p_k = \bigoplus_{l=0}^{k} \psi^l(A^{p+l}_{k-l} \cap \ker\phi).$$

Proof. Note that one can deduce easily, by induction, that for any $l \in \mathbb{N}$ and for any $a \in A_k^p$

$$\begin{aligned} \phi^l \circ \psi(a) - \psi \circ \phi^l(a) &= l(p-k+l-1)\phi^{l-1}(a), && (18)\\ \psi^l \circ \phi(a) - \phi \circ \psi^l(a) &= l(k-p+l-1)\psi^{l-1}(a). && (19) \end{aligned}$$

(i) Let $a \in A_k^p$ such that $\psi(a) = 0$. From (18) and since $p - k > 0$, for any $l \geq 0$, if $\phi^l(a) = 0$ then $\phi^{l-1}(a) = 0$. Now, since $\phi^l(a) \in A_{k-l}^{p+l}$ and since $A_{-1}^{p+l} = 0$, we have that for any $l \geq k+1$ $\phi^l(a) = 0$ which implies, by induction, that $a = 0$ and hence $\psi : A_k^p \longrightarrow A_{k+1}^{p-1}$ is injective.
(ii) Suppose that $k \leq p$. We define $P_k^p : A_k^p \longrightarrow A_k^p$ as follows

$$\begin{cases} P_k^p(a) = & \displaystyle\sum_{s=0}^{k} \alpha_s \psi^s \circ \phi^s(a) \\ \alpha_0 = 1 & \text{and } \alpha_s - (s+1)(k-p-s-2)\alpha_{s+1} = 0 \quad \text{for } 1 \leq s \leq k-1. \end{cases}$$

P_k^p satisfies

$$P_k^p \circ P_k^p = P_k^p, \quad \ker P_k^p = \psi(A_{k-1}^{p+1}) \quad \text{and} \quad ImP_k^p = A_k^p \cap \ker \phi.$$

Indeed, let $a \in A_{k-1}^{p+1}$. We have

$$\begin{aligned} P_k^p(\psi(a)) &= \sum_{s=0}^{k} \alpha_s \psi^s \circ \phi^s(\psi(a)) \\ &\overset{(18)}{=} \sum_{s=0}^{k} \alpha_s \psi^{s+1} \circ \phi^s(a) + \sum_{s=0}^{k} s(p-k+s+1)\alpha_s \psi^s \circ \phi^{s-1}(a) \\ &\overset{\phi^k(a)=0}{=} \sum_{s=0}^{k-1} \alpha_s \psi^{s+1} \circ \phi^s(a) + \sum_{s=1}^{k} s(p-k+s+1)\alpha_s \psi^s \circ \phi^{s-1}(a) \\ &= \sum_{s=0}^{k-1} (\alpha_s + (s+1)(p-k+s+2)\alpha_{s+1})\psi^{s+1} \circ \phi^s(a) \\ &= 0. \end{aligned}$$

Conversely, since $P_k^p(a) = a + \sum_{s=1}^{k} \alpha_s \psi^s \circ \phi^s(a)$, we deduce that $P_k^p(a) = 0$ implies that $a \in \psi(A_{k-1}^{p+1})$, so we have shown that $\ker P_k^p = \psi(A_{k-1}^{p+1})$. The relation $P_k^p \circ P_k^p = P_k^p$ is a consequence of the definition of P_k^p and $P_k^p \circ \psi = 0$.

Note that $\phi(a) = 0$ implies that $P_k^p(a) = a$ and hence $A_k^p \cap \ker\phi \subset ImP_k^p$. Conversely, let $a \in A_k^p$, we have

$$\begin{aligned}
\phi \circ P_k^p(a) &= \sum_{s=0}^{k} \alpha_s \phi \circ \psi^s \circ \phi^s(a) \\
&\overset{(19)}{=} \sum_{s=0}^{k} \alpha_s \psi^s \circ \phi^{s+1}(a) - \sum_{s=0}^{k} \alpha_s s(k-p-s-1)\psi^{s-1} \circ \phi^s(a) \\
&\overset{\phi^{k+1}(a)=0}{=} \sum_{s=0}^{k-1} \alpha_s \psi^s \circ \phi^{s+1}(a) - \sum_{s=1}^{k} \alpha_s s(k-p-s-1)\psi^{s-1} \circ \phi^s(a) \\
&= \sum_{s=0}^{k-1} (\alpha_s - (s+1)(k-p-s-2)\alpha_{s+1})\psi^s \circ \phi^{s+1}(a) \\
&= 0.
\end{aligned}$$

We conclude that P_k^p is a projector, $\ker P_k^p = \psi(A_{k-1}^{p+1})$ and $A_k^p \cap \ker\phi = ImP_k^p$ and we deduce immediately that $A_k^p = \psi(A_{k-1}^{p+1}) \oplus A_k^p \cap \ker\phi$. The same decomposition holds for A_{k-1}^{p+1}, and since $\psi : A_{k-1}^{p+1} \longrightarrow A_k^p$ is injective, we get

$$A_k^p = \psi \circ \psi(A_{k-2}^{p+2}) \oplus \psi(A_{k-1}^{p+1} \cap \ker\phi) \oplus A_k^p \cap \ker\phi.$$

We proceed by induction and we get the desired decomposition.
□

Let us apply this lemma to the operators (i_W, δ_h^*) and $(i_{\overline{W}}, \overline{\delta_h^*})$ acting on the spaces $T_{k,l}^{p,q}$.
Indeed, from Lemma 3.2 and (17) we deduce that, for q, l fixed, the spaces $(T_{k,l}^{p,q})_{k,p}$ and the operators (i_W, δ_h^*) satisfy the hypothesis of Lemma 3.3. Thus we have

$$T_{k,l}^{p,q} = \begin{cases} \bigoplus_{r=0}^{k} (i_W)^r \left(T_{k-r,l}^{p+r,q} \cap \ker\delta_h^*\right) & \text{if} \quad k \le p, \\ \bigoplus_{r=0}^{p} (\delta_h^*)^r \left(T_{k+r,l}^{p-r,q} \cap \ker i_W\right) & \text{if} \quad k \ge p. \end{cases}$$

On other hand, Lemma 3.2 and (3) imply that the operators $\overline{\delta_h^*}$ and $i_{\overline{W}}$ commute with δ_h^* and i_W. Hence, by Lemma 3.2 and (17), for p, q, r fixed, $((i_W)^r \left(T_{k-r,l}^{p+r,q} \cap \ker\delta_h^*\right), \overline{\delta_h^*}, i_{\overline{W}})_{q,l}$ and $((\delta_h^*)^r \left(T_{k+r,l}^{p-r,q} \cap \ker i_W\right), \overline{\delta_h^*}, i_{\overline{W}})_{q,l}$ satisfy also the hypothesis of Lemma 3.3. Thus we get the following direct sum

decompositions of $T^{p,q}_{k,l}$.

$$T^{p,q}_{k,l} = \begin{cases} \bigoplus\limits_{\substack{r=0,\dots,k \\ s=0,\dots,l}} (i_{\overline{W}})^s \circ (i_W)^r \left(T^{p+r,q+s}_{k-r,l-s} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h} \right) & \text{if} \quad k \leq p, l \leq q, \\ \bigoplus\limits_{\substack{r=0,\dots,k \\ s=0,\dots,q}} (\overline{\delta^*_h})^s \circ (i_W)^r \left(T^{p+r,q-s}_{k-r,l+s} \cap \ker \delta^*_h \cap \ker i_{\overline{W}} \right) & \text{if} \quad k \leq p, q \leq l, \\ \bigoplus\limits_{\substack{r=0,\dots,p \\ s=0,\dots,l}} (i_{\overline{W}})^s \circ (\delta^*_h)^r \left(T^{p-r,q+s}_{k+r,l-s} \cap \ker i_W \cap \ker \overline{\delta^*_h} \right) & \text{if} \quad p \leq k, l \leq q, \\ \bigoplus\limits_{\substack{r=0,\dots,p \\ s=0,\dots,q}} (\overline{\delta^*_h})^s \circ (\delta^*_h)^r \left(T^{p-r,q-s}_{k+r,l+s} \cap \ker i_W \cap \ker i_{\overline{W}} \right) & \text{if} \quad p \leq k, q \leq l. \end{cases} \tag{20}$$

Note that one can deduce easily, by applying twice the first relation in Lemma 3.3 (ii), that if $k \leq p$ and $l \leq q$ then

$$\dim \left(T^{p,q}_{k,l} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h} \right) = \dim T^{p,q}_{k,l} + \dim T^{p+1,q+1}_{k-1,l-1} - \dim T^{p+1,q}_{k-1,l} - \dim T^{p,q+1}_{k,l-1}. \tag{21}$$

Note also that since the operators δ^*_h and i_W and the operators $\overline{\delta^*_h}$ and $i_{\overline{W}}$ play symmetric roles, we have

$$\begin{aligned} \dim(T^{p,q}_{k,l} \cap \ker \delta^*_h \cap \ker i_{\overline{W}}) &= \dim(T^{p,l}_{k,q} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}), \quad k \leq p, q \leq l \\ \dim(T^{p,q}_{k,l} \cap \ker i_W \cap \ker \overline{\delta^*_h}) &= \dim(T^{k,q}_{p,l} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}), \quad p \leq k, l \leq q \\ \dim(T^{p,q}_{k,l} \cap \ker i_W \cap \ker i_{\overline{W}}) &= \dim(T^{k,l}_{p,q} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}), \quad p \leq k, q \leq l. \end{aligned} \tag{22}$$

The next step is to show that the image by ϕ of the spaces composing the direct sum decompositions (20) are actually eigenspaces of $\Delta_{P^n(\mathbb{C})}$. For simplicity, we put

$$\begin{aligned} E^{p,q}_{k,l,m}(W^r, \overline{W}^s) &= <\,,\,>^m \odot \left((i_{\overline{W}})^s \circ (i_W)^r \left(T^{p+r,q+s}_{k-r,l-s} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h} \right) \right), \\ E^{p,q}_{k,l,m}(W^r, (\overline{\delta^*_h})^s) &= <\,,\,>^m \odot \left((\overline{\delta^*_h})^s \circ (i_W)^r \left(T^{p+r,q-s}_{k-r,l+s} \cap \ker \delta^*_h \cap \ker i_{\overline{W}} \right) \right), \\ E^{p,q}_{k,l,m}((\delta^*_h)^r, \overline{W}^s) &= <\,,\,>^m \odot \left((i_{\overline{W}})^s \circ (\delta^*_h)^r \left(T^{p-r,q+s}_{k+r,l-s} \cap \ker i_W \cap \ker \overline{\delta^*_h} \right) \right), \\ E^{p,q}_{k,l,m}((\delta^*_h)^r, (\overline{\delta^*_h})^s) &= <\,,\,>^m \odot \left((\overline{\delta^*_h})^s \circ (\delta^*_h)^r \left(T^{p-r,q-s}_{k+r,l+s} \cap \ker i_W \cap \ker i_{\overline{W}} \right) \right). \end{aligned}$$

Lemma 3.4 *Let $k+p=q+l$ and $T \in E^{p,q}_{k,l,m}(W^r, \overline{W}^s) \cup E^{p,q}_{k,l,m}(W^r, (\overline{\delta^*_h})^s) \cup E^{p,q}_{k,l,m}((\delta^*_h)^r, \overline{W}^s) \cup E^{p,q}_{k,l,m}((\delta^*_h)^r, (\overline{\delta^*_h})^s)$. Then*

$$\begin{aligned}\Delta_{P^n(\mathbb{C})}\phi(T) &= 4\Big((p+k)(n-q+k)+p(p-1)+q(q-1)+r(r+1)+s(s+1)\\ &\quad +r|p-k|+s|q-l|+\frac{1}{2}\left(|p-k|+|q-l|+p-k+q-l\right)\Big)\phi(T).\end{aligned}$$

Proof. Remark that, since $\phi(<\ ,\ > \odot T) = g \odot \phi(T)$ and by (2), it suffices to show the lemma for $T \in (i_{\overline{W}})^s \circ (i_W)^r \left(T^{p+r,q+s}_{k-r,l-s} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}\right) \cup (\overline{\delta^*_h})^s \circ (i_W)^r \left(T^{p+r,q-s}_{k-r,l+s} \cap \ker \delta^*_h \cap \ker i_{\overline{W}}\right) \cup (i_{\overline{W}})^s \circ (\delta^*_h)^r \left(T^{p-r,q+s}_{k+r,l-s} \cap \ker i_W \cap \ker \overline{\delta^*_h}\right) \cup (\overline{\delta^*_h})^s \circ (\delta^*_h)^r \left(T^{p-r,q-s}_{k+r,l+s} \cap \ker i_W \cap \ker i_{\overline{W}}\right)$.
Now, since $\Delta_{\mathbb{C}^{n+1}} T = 0$, we will deduce the lemma by computing the right side of the formula composing Theorem 3.1.
First note that $\mathrm{Tr} T = 0$, $L_{\overrightarrow{r}} T = (p+q+k+l)T = 2(p+k)T$ and $T^{J_0} = \frac{1}{2}((p+q)-(p-q)^2)T$. Thus

$$\begin{aligned}&2(p+q)(p+q-1)T + 2(n-p-q)L_{\overrightarrow{r}}T + L_{\overrightarrow{r}} \circ L_{\overrightarrow{r}} T - 4T^{J_0} =\\ &\qquad (4(p+k)(n-q+k)+4(p(p-1)+q(q-1)))T.\end{aligned}$$

Now let us compute $\delta^*_h i_W T$ and $\overline{\delta^*_h} i_{\overline{W}} T$. The computation is based on (18) applied to the operators (δ^*_h, i_W) and $(\overline{\delta^*_h}, i_{\overline{W}})$.
Let $T = i_{\overline{W}^s} \circ i_{W^r}(T')$ with $T' \in T^{p+r,q+s}_{k-r,l-s} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}$. We have

$$\begin{aligned}-\delta^*_h \circ i_W \circ i_{\overline{W}^s} \circ i_{W^r}(T') &= -\delta^*_h \circ i_{W^{r+1}} \circ i_{\overline{W}^s}(T')\\ &\overset{(18)}{=} (r+1)(k-p-r)T.\\ -\overline{\delta^*_h} \circ i_{\overline{W}} \circ i_{\overline{W}^s} \circ i_{W^r}(T') &= -\overline{\delta^*_h} \circ i_{\overline{W}^{s+1}} \circ i_{W^r}(T')\\ &\overset{(18)}{=} (s+1)(l-q-s)T.\end{aligned}$$

In a similar fashion, we get:

1. for $T = (\overline{\delta^*_h})^s \circ i_{W^r}(T')$ with $T' \in T^{p+r,q-s}_{k-r,l+s} \cap \ker \delta^*_h \cap \ker i_{\overline{W}}$,

$$\begin{aligned}-\delta^*_h \circ i_W \circ (\overline{\delta^*_h})^s \circ i_{W^r}(T') &= (r+1)(k-p-r)T\\ -\overline{\delta^*_h} \circ i_{\overline{W}} \circ (\overline{\delta^*_h})^s \circ i_{W^r}(T') &= s(q-l-s-1)T;\end{aligned}$$

2. for $T = i_{\overline{W}^s} \circ (\delta_h^*)^r(T')$ with $T' \in T^{p-r,q+s}_{k+r,l-s} \cap \ker i_W \cap \ker \overline{\delta_h^*}$,

$$\begin{array}{rcl} -\delta_h^* \circ i_W \circ i_{\overline{W}^s} \circ (\delta_h^*)^r(T') &=& r(p-k-r-1)T. \\ -\overline{\delta_h^*} \circ i_{\overline{W}} \circ i_{\overline{W}^s} \circ (\delta_h^*)^r(T') &=& (s+1)(l-q-s)T; \end{array}$$

3. for $T = (\overline{\delta_h^*})^s \circ (\delta_h^*)^r(T')$ with $T' \in T^{p-r,q-s}_{k+r,l+s} \cap \ker i_W \cap \ker i_{\overline{W}}$,

$$\begin{array}{rcl} -\delta_h^* \circ i_W \circ (\overline{\delta_h^*})^s \circ (\delta_h^*)^r(T') &=& r(p-k-r-1)T. \\ -\overline{\delta_h^*} \circ i_{\overline{W}} \circ (\overline{\delta_h^*})^s \circ (\delta_h^*)^r(T') &=& s(q-l-s-1)T. \end{array}$$

The lemma follows by gathering together all the results above.□

It is natural now to compute the multiplicities of the eigenvalues obtained in Lemma 3.4. So, by Lemma 3.1, we need to compute the dimension of the spaces $E^{p,q}_{k,l,m}(W^r, \overline{W}^s)$, $E^{p,q}_{k,l,m}(W^r, (\overline{\delta_h^*})^s)$, $E^{p,q}_{k,l,m}((\delta_h^*)^r, \overline{W}^s)$ and $E^{p,q}_{k,l,m}((\delta_h^*)^r, (\overline{\delta_h^*})^s)$. Note first that, according to Lemma 3.3 (i), we have

$$\begin{array}{rcll} \dim E^{p,q}_{k,l,m}(W^r, \overline{W}^s) &=& T^{p+r,q+s}_{k-r,l-s} \cap \ker \delta_h^* \cap \ker \overline{\delta_h^*}, & k \le p, l \le q, \\ \dim E^{p,q}_{k,l,m}(W^r, (\overline{\delta_h^*})^s) &=& T^{p+r,q-s}_{k-r,l+s} \cap \ker \delta_h^* \cap \ker i_{\overline{W}}, & k \le p, q \le l, \\ \dim E^{p,q}_{k,l,m}((\delta_h^*)^r, \overline{W}^s) &=& T^{p-r,q+s}_{k+r,l-s} \cap \ker i_W \cap \ker \overline{\delta_h^*}, & p \le k, l \le q, \\ \dim E^{p,q}_{k,l,m}((\delta_h^*)^r, (\overline{\delta_h^*})^s) &=& T^{p-r,q-s}_{k+r,l+s} \cap \ker i_W \cap \ker i_{\overline{W}}, & p \le k, q \le l. \end{array}$$

From (22), to get the multiplicities it suffices to compute the dimension of $T^{p,q}_{k,l} \cap \ker \delta_h^* \cap \ker \overline{\delta_h^*}$ for any $k \le p$ and $l \le q$. According to (21), this will be done if one compute the dimension of $T^{p,q}_{k,l}$ for any p, q, k, l.
Since

$$\mathcal{S}H^{p,q}_{k,l} = T^{p,q}_{k,l} \oplus <\,,\,> \odot \mathcal{S}H^{p-1,q-1}_{k,l},$$

we have

$$\dim T^{p,q}_{k,l} = \dim \mathcal{S}H^{p,q}_{k,l} - \dim \mathcal{S}H^{p-1,q-1}_{k,l}. \tag{23}$$

To conclude, we need the following lemma.

Lemma 3.5 *We have*

$$\begin{array}{rcl} \dim \mathcal{S}H^{p,q}_{k,l} &=& \dim \mathcal{S}P^{p,q}_{k,l} + \dim \mathcal{S}P^{p-1,q-1}_{k-1,l-1} + \dim \mathcal{S}P^{p-1,q}_{k-1,l-2} + \dim \mathcal{S}P^{p,q-1}_{k-2,l-1} \\ && - \left(\dim \mathcal{S}P^{p-1,q}_{k,l-1} + \dim \mathcal{S}P^{p,q-1}_{k-1,l} + \dim \mathcal{S}P^{p,q}_{k-1,l-1} + \dim \mathcal{S}P^{p-1,q-1}_{k-2,l-2}\right). \end{array}$$

Proof. The relation is a consequence of (16) and the following equalities

$$
\begin{aligned}
&(W^* \odot \mathcal{S}P^{p-1,q}_{k,l-1}) \cap (\overline{W}^* \odot \mathcal{S}P^{p,q-1}_{k-1,l}) = W^* \odot \overline{W}^* \odot \mathcal{S}P^{p-1,q-1}_{k-1,l-1}.\\
&\left(W^* \odot \mathcal{S}P^{p-1,q}_{k,l-1} + \overline{W}^* \odot \mathcal{S}P^{p,q-1}_{k-1,l}\right) \cap r^2 \mathcal{S}P^{p,q}_{k-1,l-1} = r^2 \left(W^* \odot \mathcal{S}P^{p-1,q}_{k-1,l-2} + \overline{W}^* \odot \mathcal{S}P^{p,q-1}_{k-2,l-1}\right).
\end{aligned}
$$

□

Thus, from Lemma 3.5, (23) and (21), and after many simplifications, we get, for any $k \leq p$ and for any $l \leq q$,

$$
\begin{aligned}
\dim\left(T^{p,q}_{k,l} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}\right) \quad = \quad & \dim \mathcal{S}P^{p,q}_{k,l} + \dim \mathcal{S}P^{p-2,q-1}_{k,l-1} + \dim \mathcal{S}P^{p-1,q-2}_{k-1,l}\\
& + \dim \mathcal{S}P^{p-2,q-2}_{k-2,l-2} + \dim \mathcal{S}P^{p+1,q+1}_{k-1,l-1} + \dim \mathcal{S}P^{p,q+1}_{k-2,l-3}\\
& + \dim \mathcal{S}P^{p+1,q}_{k-3,l-2} + \dim \mathcal{S}P^{p-1,q-1}_{k-3,l-3} + \dim \mathcal{S}P^{p+1,q-1}_{k-2,l}\\
& + \dim \mathcal{S}P^{p,q-2}_{k-3,l-1} + \dim \mathcal{S}P^{p-1,q+1}_{k,l-2} + \dim \mathcal{S}P^{p-2,q}_{k-1,l-3}\\
& - \dim \mathcal{S}P^{p-1,q-1}_{k,l} - \dim \mathcal{S}P^{p-2,q-2}_{k-1,l-1} - \dim \mathcal{S}P^{p+1,q+1}_{k-2,l-2}\\
& - \dim \mathcal{S}P^{p,q}_{k-3,l-3} - \dim \mathcal{S}P^{p+1,q}_{k-1,l} - \dim \mathcal{S}P^{p+1,q-1}_{k-3,l-1}\\
& - \dim \mathcal{S}P^{p,q-2}_{k-2,l} - \dim \mathcal{S}P^{p-1,q-2}_{k-3,l-2} - \dim \mathcal{S}P^{p,q+1}_{k,l-1}\\
& - \dim \mathcal{S}P^{p-1,q+1}_{k-1,l-3} - \dim \mathcal{S}P^{p-2,q}_{k,l-2} - \dim \mathcal{S}P^{p-2,q-1}_{k-2,l-3}.
\end{aligned}
$$

Even if this formula involves a great deal of terms, it is surprising that after a computation using computing software and the formula

$$
\dim_{\mathbb{C}} \mathcal{S}P^{p,q}_{k,l} = \begin{pmatrix} n+k \\ k \end{pmatrix} \begin{pmatrix} n+l \\ l \end{pmatrix} \begin{pmatrix} n+p \\ p \end{pmatrix} \begin{pmatrix} n+q \\ q \end{pmatrix},
$$

where

$$
\begin{pmatrix} a \\ b \end{pmatrix} = \frac{a!}{b!(a-b)!},
$$

we get a simple expression of $\dim\left(T^{p,q}_{k,l} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}\right)$. Let us tabulate the results.

Conditions on p, q, k, l	Space	Complex dimension
$1 \leq k \leq p, 2 \leq l \leq q$ or $2 \leq k \leq p, 1 \leq l \leq q$	$T^{p,q}_{k,l} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}$	$\frac{(n+p-2)!(n+q-2)!(n+k-3)!(n+l-3)!n^3(n-1)^2(n-2)}{(n!)^4(p+1)!(q+1)!k!l!} \times$ $(p-k+1)(q-l+1)(n+k+l-2)(n+q+k-1)(n+p+l-1)\times$ $(n+p+q)$
$1 \leq p, 1 \leq q$	$T^{p,q}_{1,1} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}$	$\frac{(n+p-2)!(n+q-2)!n^2(n-2)pq(n+q)(n+p)(n+p+q)}{(n!)^2(p+1)!(q+1)!}$
$1 \leq p, 1 \leq l \leq q$	$T^{p,q}_{0,l} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}$	$\frac{(n+p-2)!(n+q-1)!(n+l-2)!n^2(n-1)(q-l+1)(n+p+l-1)(n+p+q)}{(n!)^3p!(q+1)!l!}$
$1 \leq k \leq p, 1 \leq q$	$T^{p,q}_{k,0} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}$	$\frac{(n+p-1)!(n+q-2)!(n+k-2)!n^2(n-1)(p-k+1)(n+q+k-1)(n+p+q)}{(n!)^3q!(p+1)!k!}$
$1 \leq p, 1 \leq q$	$T^{p,q}_{0,0} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}$	$\frac{(n+p-1)!(n+q-1)!n(n+p+q)}{(n!)^2p!q!}$
$0 \leq l \leq q$	$T^{0,q}_{0,l} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}$	$\frac{(n+q)!(n+l-1)!n(q-l+1)}{2(n!)^2(q+1)!l!}$
$0 \leq k \leq p$	$T^{p,0}_{k,0} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}$	$\frac{(n+p)!(n+k-1)!n(p-k+1)}{2(n!)^2(p+1)!k!}$

Table I.

We are now able to give the spectra and the eigenspaces with multiplicities of $\Delta_{P^n(\mathbb{C})}$ acting on $\mathcal{S}^{p,q}(P^n(\mathbb{C}), \mathbb{C})$. Note that the spectra of $\Delta_{P^n(\mathbb{C})}$ acting on $\mathcal{S}^{p,q}(P^n(\mathbb{C}), \mathbb{C})$ is the same as the spectra of $\Delta_{P^n(\mathbb{C})}$ acting on $\mathcal{S}^{q,p}(P^n(\mathbb{C}), \mathbb{C})$ and the eigenspaces are conjugated. So, we restrict ourself to the case $p \leq q$. Fix $p, l \in \mathbb{N}$ and consider the space $\mathcal{S}^{p,p+l}(P^n(\mathbb{C}), \mathbb{C})$. We have, from Lemma 3.1,

$$\phi : \bigoplus_{\substack{0 \leq m \leq p \\ k \in \mathbb{N}}} <\, ,\, >^m \odot T^{p-m,p+l-m}_{k+l,k} \longrightarrow \mathcal{S}^{p,p+l}(P^n(\mathbb{C}), \mathbb{C})$$

is injective and its image is dense. To obtain the eigenvalues and eigenspaces of $\Delta_{P^n(\mathbb{C})}$ acting on $\mathcal{S}^{p,p+l}(P^n(\mathbb{C}),\mathbb{C})$, we split any $T_{k+l,k}^{p-m,p+l-m}$ above according to (20) and we apply Lemma 3.4. Note also that, according to Table I, the dimension of some eigenspaces vanishes when $n=1$ or $n=2$ and so, one must distingue three cases $n\geq 3$, $n=2$ or $n=1$. To state the results, we introduce some notations.

We put

$$
\begin{aligned}
S_0 &= \Big\{(m,k,r,s)\in\mathbb{N}^4/0\leq m\leq p, 0\leq k< p-m-l, 0\leq r\leq k+l, 0\leq s\leq k\Big\},\\
S_1 &= \Big\{(m,k,r,s)\in\mathbb{N}^4/0\leq m\leq p, \max(0,p-m-l)\leq k< p-m+l, 0\leq r\leq p-m,\\
&\quad 0\leq s\leq k\big\},\\
S_2 &= \Big\{(m,k,r,s)\in\mathbb{N}^4/0\leq m\leq p, k\geq p-m+l, 0\leq r\leq p-m, 0\leq s\leq p-m+l\Big\},\\
V_{r,s,0}^{p,l,m,k} &= E_{k+l,k,m}^{p-m,p+l-m}(W^r,\overline{W}^s) \quad \text{if} \quad (m,k,r,s)\in S_0,\\
V_{r,s,1}^{p,l,m,k} &= E_{k+l,k,m}^{p-m,p+l-m}((\delta_h^*)^r,\overline{W}^s) \quad \text{if} \quad (m,k,r,s)\in S_1,\\
V_{r,s,2}^{p,l,m,k} &= E_{k+l,k,m}^{p-m,p+l-m}((\delta_h^*)^r,(\overline{\delta_h^*})^s) \quad \text{if} \quad (m,k,r,s)\in S_2.
\end{aligned}
$$

The eigenvalue obtained in Lemma 3.4 becomes

$$
\begin{aligned}
&\lambda_{r,s,n}^{p,l,m,k} = 4\,[(p-m+k+l)(n-p+m+k)+(p-m)(p-m-1)\\
&+(p-m+l)(p-m+l-1)+r(r+1)+s(s+1)+r|p-m-k-l|+s|p-m+l-k|\\
&+\tfrac{1}{2}\left(|p-m-k-l|+|p-m+l-k|+2(p-m-k)\right)\Big].
\end{aligned}
$$

Finally, the following notations are needed to treat the case $n=2$.

$$
\begin{aligned}
S_0^0 &= \Big\{(m,k,r)\in\mathbb{N}^3/0\leq m\leq p, 0\leq k< p-m-l, 0\leq r\leq k\Big\},\\
S_0^1 &= \Big\{(m,k,r)\in\mathbb{N}^3/0\leq m\leq p, 0\leq k< p-m-l, 0\leq r\leq k+l\Big\},\\
S_1^0 &= \Big\{(m,k,r)\in\mathbb{N}^3/0\leq m\leq p, \max(0,p-m-l)\leq k< p-m+l, 0\leq r\leq k\Big\},\\
S_1^1 &= \Big\{(m,k,r)\in\mathbb{N}^3/0\leq m\leq p, \max(0,p-m-l)\leq k< p-m+l, 0\leq r\leq p-m\Big\},\\
S_2^0 &= \Big\{(m,k,r)\in\mathbb{N}^3/0\leq m\leq p, k\geq p-m+l, 0\leq r\leq p-m+l\Big\},\\
S_2^1 &= \Big\{(m,k,r)\in\mathbb{N}^3/0\leq m\leq p, k\geq p-m+l, 0\leq r\leq p-m\Big\},
\end{aligned}
$$

$$
W_{r,0,0}^{p,l,m,k} = V_{k+l,r,0}^{p,l,m,k} \quad \text{if} \quad (m,k,r)\in S_0^0, \quad W_{r,0,1}^{p,l,m,k} = V_{r,k,0}^{p,l,m,k} \quad \text{if} \quad (m,k,r)\in S_0^1,
$$

$$
\begin{array}{lllll}
W^{p,l,m,k}_{r,1,0} & = & V^{p,l,m,k}_{p-m,r,1} & \text{if} \quad (m,k,r) \in S^0_1, & W^{p,l,m,k}_{r,1,1} = V^{p,l,m,k}_{r,k,1} \quad \text{if} \quad (m,k,r) \in S^1_1, \\
W^{p,l,m,k}_{r,2,0} & = & V^{p,l,m,k}_{p-m,r,2} & \text{if} \quad (m,k,r) \in S^0_2, & W^{p,l,m,k}_{r,2,1} = V^{p,l,m,k}_{r,p-m+l,2} \quad \text{if} \quad (m,k,r) \in S^1_2.
\end{array}
$$

The following theorem is an immediate consequence of Lemma 3.1, (20) and Lemma 3.4.

Theorem 3.2 *Let $n, p, l \in \mathbb{N}$ such that $n \geq 3$. Then:*

1. $\phi : \bigoplus_{i=0,\ldots,2} \left(\bigoplus_{(m,k,r,s)\in S_i} V^{p,l,m,k}_{r,s,i} \right) \longrightarrow \mathcal{S}^{p,p+l}(P^n(\mathbb{C}), \mathbb{C})$ *is injective and its image is dense,*

2. *for any $i = 0, \ldots, 2$ and for any $(m,k,r,s) \in S_i$, $\phi(V^{p,l,m,k}_{r,s,i})$ is an eigenspace of $\Delta_{P^n(\mathbb{C})}$ associated to the eigenvalue $\lambda^{p,l,m,k}_{r,s,n}$,*

3. *for any $i = 0, \ldots, 2$ and for any $(m,k,r,s) \in S_i$, the dimension of $\phi(V^{p,l,m,k}_{r,s,i})$ is given by Table I and (22) since*

$$
\begin{array}{lll}
\dim\left(\phi(V^{p,l,m,k}_{r,s,0})\right) & = & \dim\left(T^{p-m+r,p-m+l+s}_{k+l-r,k-s} \cap \ker \delta^*_h \cap \ker \overline{\delta^*_h}\right), \\
\dim\left(\phi(V^{p,l,m,k}_{r,s,1})\right) & = & \dim\left(T^{p-m-r,p-m+l+s}_{k+l+r,k-s} \cap \ker \overline{\delta^*_h} \cap \ker i_W\right), \\
\dim\left(\phi(V^{p,l,m,k}_{r,s,2})\right) & = & \dim\left(T^{p-m-r,p-m+l-s}_{k+l+r,k+s} \cap \ker i_W \cap \ker i_{\overline{W}}\right).
\end{array}
$$

By deleting in Theorem 4.2 the spaces $V^{p,l,m,k}_{r,s,i}$ whose dimension vanishes in the case $n = 2$ or $n = 1$, we get the following theorem.

Theorem 3.3 *Let $p, l \in \mathbb{N}$. Then:*

1. $\phi : \bigoplus_{i,j=0,\ldots,2} \left(\bigoplus_{(m,k,r)\in S^j_i} W^{p,l,m,k}_{r,i,j} \right) \longrightarrow \mathcal{S}^{p,p+l}(P^2(\mathbb{C}), \mathbb{C})$ *is injective and its image is dense;*

2. *the spaces $W^{p,l,m,k}_{r,0,0}$, $W^{p,l,m,k}_{r,0,1}$, $W^{p,l,m,k}_{r,1,0}$, $W^{p,l,m,k}_{r,1,1}$, $W^{p,l,m,k}_{r,2,0}$ and $W^{p,l,m,k}_{r,2,1}$ are eigenspaces of $\Delta_{P^2(\mathbb{C})}$ associated, respectively, to the eigenvalues $\lambda^{p,l,m,k}_{k+l,r,2}$, $\lambda^{p,l,m,k}_{r,k,2}$, $\lambda^{p,l,m,k}_{p-m,r,2}$, $\lambda^{p,l,m,k}_{r,k,2}$, $\lambda^{p,l,m,k}_{p-m,r,2}$, $\lambda^{p,l,m,k}_{r,p-m+l,2}$;*

3. *the dimension of* $\phi(W^{p,l,m,k}_{r,i,j})$ *is given by Table I and* (22) *since*

$$
\begin{aligned}
\dim\left(\phi(W^{p,l,m,k}_{r,0,0})\right) &= \dim\left(T^{p-m+k+l,p-m+l+r}_{0,k-r}\cap\ker\delta^*_h\cap\ker\overline{\delta^*_h}\right),\\
\dim\left(\phi(W^{p,l,m,k}_{r,0,1})\right) &= \dim\left(T^{p-m+r,p-m+l+k}_{k+l-r,0}\cap\ker\delta^*_h\cap\ker\overline{\delta^*_h}\right),\\
\dim\left(\phi(W^{p,l,m,k}_{r,1,0})\right) &= \dim\left(T^{0,p-m+l+r}_{k+l+p-m,k-r}\cap\ker\overline{\delta^*_h}\cap\ker i_W\right),\\
\dim\left(\phi(W^{p,l,m,k}_{r,1,1})\right) &= \dim\left(T^{p-m-r,p-m+l+k}_{k+l+r,0}\cap\ker\overline{\delta^*_h}\cap\ker i_W\right),\\
\dim\left(\phi(W^{p,l,m,k}_{r,2,1})\right) &= \dim\left(T^{0,p-m+l-r}_{k+l+p-m,k+r}\cap\ker i_W\cap\ker i_{\overline{W}}\right),\\
\dim\left(\phi(W^{p,l,m,k}_{r,2,2})\right) &= \dim\left(T^{p-m-r,0}_{k+l+r,k+p-m+l}\cap\ker i_W\cap\ker i_{\overline{W}}\right);
\end{aligned}
$$

4. *for* $P^1(\mathbb{C})$*, we have*

$$
\begin{aligned}
\phi:\bigoplus_{\substack{0\le m\le p\\ 0\le k<p-m-l}}\left(V^{p,l,m,k}_{k+l-1,k-1,0}\oplus V^{p,l,m,k}_{k+l,k,0}\right)\oplus\bigoplus_{\substack{0\le m\le p\\ \max(0,p-m-l)\le k<p-m+l}}\left(V^{p,l,m,k}_{p-m-1,k-1,1}\oplus V^{p,l,m,k}_{p-m,k,1}\right)\\
\oplus\bigoplus_{\substack{0\le m\le p\\ k\ge p-m+l}}\left(V^{p,l,m,k}_{p-m-1,p-m+l-1,2}\oplus V^{p,l,m,k}_{p-m,p-m+l,2}\right)\longrightarrow\mathcal{S}^{p,p+l}(P^1(\mathbb{C}),\mathbb{C})
\end{aligned}
$$

is injective and its image is dense. Moreover, the image by ϕ *of all the spaces* $V^{p,l,m,k}_{r,s,i}$ *composing the above direct sum decomposition are eigenspaces of* $\Delta_{P^1(\mathbb{C})}$ *associated to the eigenvalue* $\lambda^{p,l,m,k}_{r,s,1}$*, and their dimensions can be deduced from 3. Theorem 4.2.*

Remark 3.1 *In [11], Ikeda and Taniguchi computed the eigenvalues of* $\Delta_{P^n(\mathbb{C})}$ *acting on* $\Omega(P^n(\mathbb{C}),\mathbb{C})$ *and determined the spaces of eigenforms as representation spaces, but they did not give the multiplicities. The formula obtained in Theorem 2.1 can be used in the case of differential forms to show that the images by* ϕ *of the spaces composing the direct sum decomposition (7.2) in [11] are eigenspaces of* $\Delta_{P^n(\mathbb{C})}$*. The dimensions of these spaces can be computed in a similar way as in [12]. Unfortunately, the formulas obtained are much more complicated than the case of the spheres. However, in [11, Theorem 7.13], Ikeda and Taniguchi showed that these spaces are irreducible* $SU(n+1)$*-modules and they computed their highest weights. Hence, may be, one can use the Weyl dimension formula to compute the dimensions of these spaces and to get the multiplicities of the eigenvalues.*

Finally, we apply Theorems 3.2 and 3.3 for the low values of p and l and we tabulate the results.

Spaces $n \geq 1$	Eigenvalues $k \in \mathbb{N}$	Eigenspaces	Complex dimension
$C^\infty(P^n(\mathbb{C}))$	$4k(n+k)$	$\phi\left(T^{0,0}_{k,k}\right)$	$\frac{n(n+2k)((n+k-1)!)^2}{(n!)^2(k!)^2}$
$\mathcal{S}^{0,1}(P^n(\mathbb{C}),\ \mathbb{C})$	$4(n+1)$	$T^{0,1}_{1,0}$	$n(n+2)$
	$4(k+1)(n+k+2)$	$\phi\circ\overline{\delta^*_h}\left(T^{0,0}_{k+2,k+2}\right)$	$\frac{n(n+2k+4)((n+k+1)!)^2}{(n!)^2((k+2)!)^2}$
	$4(k+2)(n+k+1)$	$\phi\left(T^{0,1}_{k+2,k+1}\cap\ker i_{\overline{W}}\right)$	$\frac{(k+1)n(n-1)(n+k+2)(n+2k+3)((n+k)!)^2}{(n!)^2((k+2)!)^2}$
$\mathcal{S}^{1,0}(P^n(\mathbb{C}),\ \mathbb{C})$	$4(n+1)$	$\phi\left(T^{1,0}_{0,1}\right)$	$n(n+2)$
	$4(k+2)(n+k+2)$	$\phi\circ\delta^*_h\left(T^{0,0}_{k+2,k+2}\right)$	$\frac{n(n+2k+4)((n+k+1)!)^2}{(n!)^2((k+2)!)^2}$
	$4(k+2)(n+k+1)$	$\phi\left(T^{1,0}_{k+1,k+2}\cap\ker i_W\right)$	$\frac{(k+1)n(n-1)(n+k+2)(n+2k+3)((n+k)!)^2}{(n!)^2((k+2)!)^2}$

Table II.

Spaces $n \geq 1$	Eigenvalues $k \in \mathbb{N}$	Eigenspaces	Complex dimension
$\mathcal{S}^{0,2}(P^n(\mathbb{C}),\ \mathbb{C})$	$8(n+2)$	$\phi\left(T^{0,2}_{2,0}\cap\ker\overline{\delta^*_h}\right)$	$\frac{n(n+4)(n+1)^2}{4}$
	$12(n+3)$	$\phi\circ i_{\overline{W}}\left(T^{0,3}_{3,0}\cap\ker\overline{\delta^*_h}\right)$	$\frac{n(n+1)^2(n+2)^2(n+6)}{36}$
	$4(k+4)(n+k+4)$	$\phi\circ(\overline{\delta^*_h})^2\left(T^{0,0}_{k+4,k+4}\right)$	$\frac{((n+k+3)!)^2 n(n+2k+8)}{(n!)^2((k+4)!)^2}$
	$12(n+2)$	$\phi\left(T^{0,2}_{3,1}\cap\ker\overline{\delta^*_h}\right)$	$\frac{n(n+1)^2(n-1)(n+2)(n+5)}{9}$
	$4(k+4)(n+k+3)$	$\phi\circ\overline{\delta^*_h}\left(T^{0,1}_{k+4,k+3}\cap\ker i_{\overline{W}}\right)$	$\frac{((n+k+2)!)^2 n(n-1)(k+3)(n+k+4)(n+2k+7)}{(n!)^2((k+4)!)^2}$
	$4(k^2+(n+6)k+4n+10)$	$\phi\left(T^{0,2}_{k+4,k+2}\cap\ker i_{\overline{W}}\right)$	$\frac{(n+k+2)!(n+k+1)!n^2(n-1)(k+1)(n+k+5)(n+2k+6)}{2(n!)^2(k+4)!(k+3)!}$

Table III.

Spaces $n \geq 1$	Eigenvalues $k \in \mathbb{N}$	Eigenspaces	Complex dimension
$\mathcal{S}^{2,0}(P^n(\mathbb{C}),\ \mathbb{C})$	$8(n+2)$	$\phi\left(T^{2,0}_{0,2}\cap\ker\delta^*_h\right)$	$\frac{n(n+4)(n+1)^2}{4}$
	$12(n+3)$	$\phi\circ i_W\left(T^{3,0}_{0,3}\cap\ker\delta^*_h\right)$	$\frac{n(n+1)^2(n+2)^2(n+6)}{36}$
	$4(k+4)(n+k+4)$	$\phi\circ(\delta^*_h)^2\left(T^{0,0}_{k+4,k+4}\right)$	$\frac{((n+k+3)!)^2 n(n+2k+8)}{(n!)^2((k+4)!)^2}$
	$12(n+2)$	$\phi\left(T^{2,0}_{1,3}\cap\ker\delta^*_h\right)$	$\frac{n(n+1)^2(n-1)(n+2)(n+5)}{9}$
	$4(k+4)(n+k+3)$	$\phi\circ\delta^*_h\left(T^{1,0}_{k+3,k+4}\cap\ker i_W\right)$	$\frac{((n+k+2)!)^2 n(n-1)(k+3)(n+k+4)(n+2k+7)}{(n!)^2((k+4)!)^2}$
	$4(k^2+(n+6)k+4n+10)$	$\phi\left(T^{2,0}_{k+2,k+4}\cap\ker i_W\right)$	$\frac{(n+k+2)!(n+k+1)!n^2(n-1)(k+1)(n+k+5)(n+2k+6)}{2(n!)^2(k+4)!(k+3)!}$

Table IV.

Spaces $n \geq 1$	Eigenvalues $k \in \mathbb{N}$	Eigenspaces	Complex dimension
$\mathcal{S}^{1,1}(P^n(\mathbb{C}), \mathbb{C})$	$4(n+1)$	$\phi\left(T^{1,1}_{0,0}\right)$	$n(n+2)$
	$4(k+2)(n+k+2)$	$\phi \circ \delta^*_h \circ \overline{\delta^*_h}\left(T^{0,0}_{k+2,k+2}\right)$	$\frac{((n+k+1)!)^2 n(n+2k+4)}{(n!)^2((k+2)!)^2}$
	$4k(n+k)$	$\phi\left(<\, ,\, > \odot T^{0,0}_{k,k}\right)$	$\frac{((n+k-1)!)^2 n(n+2k)}{(n!)^2(k!)^2}$
	$4(k+2)(n+k+1)$	$\phi \circ \overline{\delta^*_h}\left(T^{1,0}_{k+1,k+2} \cap \ker i_W\right)$ $\oplus \phi \circ \delta^*_h\left(T^{0,1}_{k+2,k+1} \cap \ker i_{\overline{W}}\right)$	$\frac{2((n+k)!)^2 n(n-1)(k+1)(n+k+1)(n+2k+3)}{(n!)^2((k+2)!)^2}$
	$4(k+2)(n+k)$	$\phi\left(T^{1,1}_{k+1,k+1} \cap \ker i_W \cap \ker i_{\overline{W}}\right)$	$\frac{((n+k-1)!)^2 n^2(n-2)(k+1)^2(n+k+1)^2(n+2k+2)}{(n!)^2((k+2)!)^2}$

Table V.

By setting $n = 2$ in Tables III-V, we get the eigenvalues and the eigenspaces of $\Delta_{P^2(\mathbb{C})}$ acting on $\mathcal{S}^2(P^2(\mathbb{C}), \mathbb{C})$. These results complete the results obtained in [23] since we give explicitly the eigenspaces. Note that there is a misprint in [23, Table 1 pp. 227]. The degeneracy of $\frac{2}{3}\Lambda(m+1)(m+3)$ is, actually, $2(m+2)^3$ (this is the value obtained by Warner in [23,(6.5)]).

Spaces	Eigenvalues $m \in \mathbb{N}$	Eigenspaces	Complex dimension
$\mathcal{S}^{0,2}(P^2(\mathbb{C}), \mathbb{C})$	32	$\phi\left(T^{0,2}_{2,0} \cap \ker \overline{\delta^*_h}\right)$	27
	60	$\phi \circ i_{\overline{W}}\left(T^{0,3}_{3,0} \cap \ker \overline{\delta^*_h}\right)$	64
	$4(m+4)(m+6)$	$\phi \circ (\overline{\delta^*_h})^2\left(T^{0,0}_{m+4,m+4}\right)$	$(m+5)^3$
	48	$\phi\left(T^{0,2}_{3,1} \cap \ker \overline{\delta^*_h}\right)$	56
	$4(m+4)(m+5)$	$\phi \circ \overline{\delta^*_h}\left(T^{0,1}_{m+4,m+3} \cap \ker i_{\overline{W}}\right)$	$\frac{(m+3)(m+6)(2m+9)}{2}$
	$4(m^2+8m+18)$	$\phi\left(T^{0,2}_{m+4,m+2} \cap \ker i_{\overline{W}}\right)$	$(m+1)(m+7)(m+4)$

Table VI.

Spaces	Eigenvalues $m \in \mathbb{N}$	Eigenspaces	Complex dimension
$\mathcal{S}^{2,0}(P^2(\mathbb{C}), \mathbb{C})$	32	$\phi\left(T^{2,0}_{0,2} \cap \ker \delta^*_h\right)$	27
	60	$\phi \circ i_W\left(T^{3,0}_{0,3} \cap \ker \delta^*_h\right)$	64
	$4(m+4)(m+6)$	$\phi \circ (\delta^*_h)^2\left(T^{0,0}_{m+4,m+4}\right)$	$(m+5)^3$
	48	$\phi\left(T^{2,0}_{1,3} \cap \ker \delta^*_h\right)$	56
	$4(m+4)(m+5)$	$\phi \circ \delta^*_h\left(T^{1,0}_{m+3,m+4} \cap \ker i_W\right)$	$\frac{(m+3)(m+6)(2m+9)}{2}$
	$4(m^2+8m+18)$	$\phi\left(T^{2,0}_{m+2,m+4} \cap \ker i_W\right)$	$(m+1)(m+7)(m+4)$

Table VII.

Spaces	Eigenvalues $m \in \mathbb{N}$	Eigenspaces	Complex dimension
$\mathcal{S}^{1,1}(P^2(\mathbb{C}), \mathbb{C})$	12	$\phi\left(T^{1,1}_{0,0}\right)$	8
	$4(m+2)(m+4)$	$\phi \circ \delta^*_h \circ \overline{\delta^*_h}\left(T^{0,0}_{m+2,m+2}\right)$	$(m+3)^3$
	$4m(m+2)$	$\phi\left(< , > \odot T^{0,0}_{m,m}\right)$	$(m+1)^3$
	$4(m+2)(m+3)$	$\phi \circ \overline{\delta^*_h}\left(T^{1,0}_{m+1,m+2} \cap \ker i_W\right)$	$(m+1)(m+3)(2m+5)$

Table VIII.

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Mohamed Boucetta
Faculté des Sciences et Techniques
BP 549 Marrakech, Morocco.
Email: *mboucetta2@yahoo.fr*